LCR Discharge Circuit Analysis

The following is a derivation of current as a function of time in the switched LCR circuit shown at Figure 1 below.

The reasons for this analysis are to:

1. enable you to estimate the performance of your capacitive discharge welder,
2. allow you to optimise your welder design and parameters for a given weld,
3. allow me to revise my somewhat rusty undergraduate mathematics.

If mathematics and circuit theory are not your cup-of-tea then, with an understanding of what constitutes under, critical and over damping you can use the results at equations [31], [35] and [40] directly. However the accompanying spreadsheet will allow you to apply these calculations (and more) directly to your welder design with no mathematics at all.

As you might expect, this modelling exercise has a number of limitations:

1. A model is an attempt to simulate actual circuit behaviour but it is still just a model. Because the model uses simplifications and assumptions about your real welder you can expect variance between what your model predicts and what you might actually measure. Your real circuit measurements are absolute but be aware that you may not actually be measuring what you think you are measuring.
2. The model switch-on speed is instantaneous. In reality switching times for your welder will take at several hundred nanoseconds with a well-designed MOSFET drive with high speed MOSFETs in the main switch.
3. Turn off transients have not been calculated. If you are actively turning off you main MOSFET switch at high speed and interrupting significant currents then you can expect inductive voltage spikes of many thousands of volts. These MUST be suppressed using fast high-current free wheeling diodes, appropriately rated over-voltage protection such as Transient Voltage Suppression (TVS), or other appropriate forms of over-voltage protection. If you do not heed this warning then you risk destroying your primary switch MOSFETs.
4. This model is based on lumped components. In reality the inductance, capacitance and resistance of your discharge circuit will be distributed. A general hand-calculation for the performance of distributed components is extremely complicated and is probably of little value. Even computer modelling of distributed components is at best an approximation of actual performance.
5. I have made no allowances for component changes under discharge conditions. At relatively high currents even large diameter copper conductors will heat and their resistance will increase as the discharge proceeds. These effects can be quite pronounced even with

discharge pulses of just a few milliseconds. They will be compounding for short duty cycles repetitive pulses.

6. The values of the inductance, capacitance and resistance for your circuit can only be estimated because these will change with age, temperature, frequency and the physical layout of your weld cables.

The analysis is largely completed from first principles. This level of analysis is typically taught at advanced undergraduate (college) level and is probably better explained in your favourite electrical engineering or advanced level mathematics textbook. The same results can be obtain with more elegance and significantly less mathematics in the complex frequency domain using Laplace transforms - but this requires an understanding of Laplace transforms (another topic that will almost certainly be covered in your favourite textbook).

The results have been validated by computational modelling, a comparison of the model with actual circuit performance, and a thorough review of the mathematics.

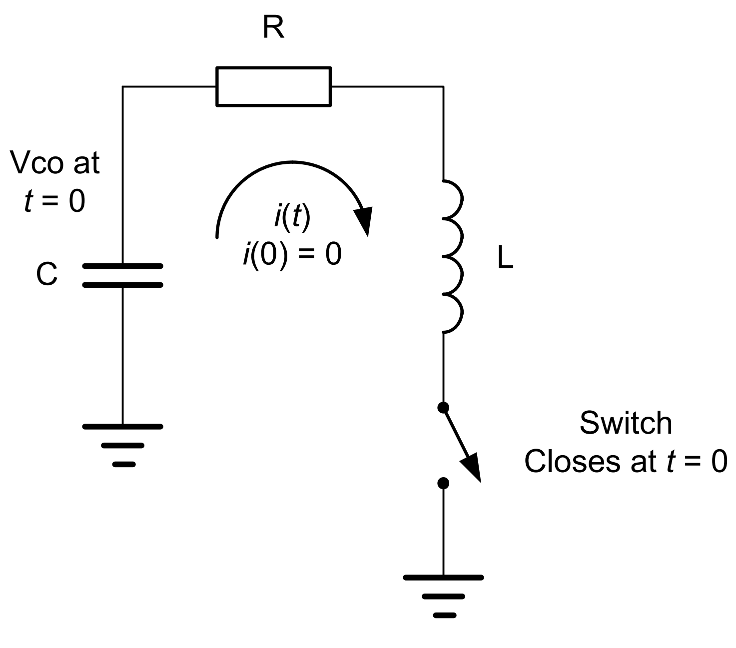

Figure 1. RCL Circuit

Figure 1 shows a simple lumped-component circuit comprising a capacitor charged to an arbitrary initial voltage, V_{c0}, an inductor with no initial current flowing and a resistor. At time *t* = 0 the switch is closed. The following calculations develop expressions for the current flowing in the circuit as a function of time, $i(t)$.

Applying Kirchhoff's Voltage Law (KVL) and noting that signs will sort themselves out in the analysis we have:

$$Vr + Vl + Vc = 0 \qquad [1]$$

Ohm's Law gives us the voltage at any instant across the resistor:

$$Vr = R\, i(t) \qquad [2]$$

We also have fundamental relationships between current and voltage for reactive (capacitive and inductive) components:

$$Vl = -L\,\frac{di(t)}{dt} \qquad [3]$$

$$i(t) = C\,\frac{dVc(t)}{dt} \qquad [4]$$

Rearranging [4]:

$$\frac{dVc(t)}{dt} = \frac{i(t)}{C} \qquad [5]$$

Substitution [2] and [3] into [1]:

$$R\,i(t) - L\,\frac{di(t)}{dt} + Vc = 0 \qquad [6]$$

Differentiating [5] with respect to *t*:

$$R\frac{di(t)}{dt} - L\,\frac{d^2i(t)}{dt^2} + \frac{dVc}{dt} = 0 \qquad [7]$$

Substituting [5] into [7]:

$$R\frac{di(t)}{dt} - L\,\frac{d^2i(t)}{dt^2} + \frac{i(t)}{C} = 0 \qquad [8]$$

Rearranging [8]:

$$-L\,\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{i(t)}{C} = 0 \qquad [9]$$

Dividing [9] by -*L* we recognise this as a linear second order differential equation:

$$\frac{d^2i(t)}{dt^2} - \frac{di(t)}{dt}\frac{R}{L} + \ i(t)\,\frac{1}{LC} = 0 \qquad [10]$$

A possible trail solution to [10] is:

$$i(t) = Ke^{st} \qquad [11]$$

where *s* can be real, imaginary or complex.

Substituting [11] into [10] and completing the differentials:

$$Ke^{st}\left[s^2 - s\frac{R}{L} + \frac{1}{LC}\right] = 0 \qquad [12]$$

The solution *K* = 0 is trivial and e^{st}cannot be 0 for all *t* so the quadratic (the term in brackets) must necessarily = 0 for all possible values of *t*. Using the standard solution for the roots of a quadratic equation:

$as^2 + bs + \ c = 0$ we can solve for *s*: [13]

$$s \ = \ \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad [14]$$

The two possible solutions are:

$$s_1,\ s_2 \ = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4\frac{1}{LC}}}{2}$$

which simplify to:

$$s_1,\ s_2 \ = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad [15]$$

However we are seeking a general solution to [10] and there are two possible values for *s* and we have two initial conditions that must be satisfied for current and voltage (these are known as boundary conditions).

So the general solution must be of the form:

$$i(t) = K_1 e^{S_1 t} + \ K_2 e^{S_2 t} \qquad [16]$$

Solving for K_1 and K_2:

The initial current at *t* = 0 is $i(0) = 0$. So from [16], and remembering that e^0 = 1, we have:

$$0 = K_1 + \ K_2 \qquad [17]$$

Rearranging [17]:

$$K_1 = \ -K_2 \qquad [18]$$

The initial constraint $i(0) = 0$ also requires from [2] that Vr = 0. It follows from [1] that:

$$Vl + Vc = 0 \text{ at } t = 0 \qquad [19]$$

Let V_{c0} be the initial voltage on the capacitor at *t* = 0. Rearranging [19]:

$$-Vl = V_{c0} \qquad [20]$$

Substituting [3] into [20] for Vl and noting the change in sign due to the double negative:

$$L \ \frac{di(t)}{dt} = V_{c0} \text{ at } t = 0 \qquad [21]$$

Substituting [16] into [21] and completing the differential gives:

$$L \ [K_1 S_1 e^{S_1 t} + \ K_2 S_2 e^{S_2 t}] = V_{c0} \text{ at } t = 0 \qquad [22]$$

Remembering the identity e^0 = 1, this simplifies to:

$$K_1 S_1 + \ K_2 S_2 = \frac{V_{c0}}{L} \qquad [23]$$

Substituting [18] for K_2 in 22 gives us:

$$K_1 S_1 - \ K_1 S_2 = \frac{V_{c0}}{L} \qquad [24]$$

Rearranging [24] to solve for K_1:

$$K_1 = \frac{V_{c0}}{L}\frac{1}{S_1 - S_2} \qquad [25]$$

and from [18] the solution for K_2 is:

$$K_2 = -\frac{V_{c0}}{L}\frac{1}{S_1 - S_2} \qquad [26]$$

Substituting [15], [25] and [26] into the general solution [16] gives us:

$$i(t) = \frac{V_{c0}}{L}\frac{1}{\left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) - \left(-\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)}\left[e^{\left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)t} - e^{\left(-\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)t}\right]$$

which simplifies to:

$$i(t) = \frac{V_{c0}}{L}\frac{1}{2\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}}\left[e^{-\left(\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)t} - e^{-\left(\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)t}\right] \qquad [27]$$

We can simplify this expression further by incorporating the conventional constants:

$w_0 = \frac{1}{\sqrt{LC}}$ a measure of angular frequency in radians per second [28]

$\varsigma = \frac{R}{2}\sqrt{\frac{C}{L}}$ referred to as the damping factor (unit-less) [29]

and noting that:

$$w_0\varsigma = \frac{R}{2L} \quad \text{and} \quad w_0 2L = \frac{R}{\varsigma} \qquad [30]$$

[27] simplifies to:

$$i(t) = \frac{V_{c0}}{L}\frac{1}{2\sqrt{{w_0}^2\varsigma^2 - {w_0}^2}}\left[e^{-\left(w_0\varsigma - \sqrt{(w_0\varsigma)^2 - {w_0}^2}\right)t} - e^{-\left(w_0\varsigma + \sqrt{(w_0\varsigma)^2 - {w_0}^2}\right)t}\right]$$

$$\boxed{i(t) = \frac{V_{c0}}{R}\frac{\varsigma}{\sqrt{\varsigma^2 - 1}}\left[e^{-w_0\left(\varsigma - \sqrt{\varsigma^2 - 1}\right)t} - e^{-w_0\left(\varsigma + \sqrt{\varsigma^2 - 1}\right)t}\right]} \qquad [31]$$

This the final form of the general solution for current in an LCR discharge circuit.

We can see from [16] and [31] that our solutions for s are now of the form:

$$s_1,\ s_2 = -w_0\left(\varsigma \pm \sqrt{\varsigma^2 - 1}\right) \qquad [32]$$

This leads to three specific results from our general solution [31] depending on the value of the damping factor, ς.

When ς is:

a. greater than 1 then s_1 and s_2 at [32], are unequal real numbers and the circuit is considered to be overdamped.

b. equal to 1 then s_1 and s_2 are equal real numbers and the circuit is considered to be critically damped.

c. less than 1 then s_1 and s_2 are complex conjugates and the circuit is considered to be underdamped (and will exhibit oscillatory behaviour).

The current in the overdamped condition can be calculated directly using the general solution [31] however, the critically damped and under-damped classes require further consideration because they result in indeterminate results and complex numbers when applied to the general solution in this form.

Critically Damped Solution

If we assign ς = 1, the condition for critical damping, then our general solution [31] reduces to:

$$i(t) = \frac{0}{0}$$

This indeterminate form can be solved by applying L'Hospital's rule. As a first step we need to identify the elements of the characteristic equation [32] that lead to the indeterminate result. If we substitute $\varsigma = 1$ in [32] in the characteristic equation except for the terms $\sqrt{\varsigma^2 - 1}$ which result in the zeros then we have:

$$i(t) = \frac{V_{c0}}{R}\frac{1}{\sqrt{\varsigma^2-1}}\left[e^{-w_0\left(1-\sqrt{\varsigma^2-1}\right)t} - e^{-w_0\left(1+\sqrt{\varsigma^2-1}\right)t}\right] \qquad [33]$$

We can rearrange the exponentials in the brackets using the identity $e^{a+b} = e^a e^b$ as follows:

$$e^{-w_0\left(1-\sqrt{\varsigma^2-1}\right)t} - e^{-w_0\left(1+\sqrt{\varsigma^2-1}\right)t} = e^{-w_0 t}\left(e^{w_0 t\sqrt{\varsigma^2-1}} - e^{-w_0 t\sqrt{\varsigma^2-1}}\right)$$ so that

$$i(t) = \frac{V_{c0}}{R}\, e^{-w_0 t}\left[\frac{e^{w_0 t\sqrt{\varsigma^2-1}} - e^{-w_0 t\sqrt{\varsigma^2-1}}}{\sqrt{\varsigma^2-1}}\right] \qquad [34]$$

The bracketed term in [34] identifies the elements that are causing the indeterminate answer because as $\varsigma \to 1$, $\sqrt{\varsigma^2 - 1} \to 0$. L'Hospital's rule states that under certain criteria (which are met in this instance) we can determine the limit of one function divided by another by examining the differentials of these functions, that is to say:

$$Limit_{x\to c}\frac{f(x)}{g(x)} = Limit_{x\to c}\frac{f\prime(x)}{g\prime(x)}$$

Let $\sqrt{\varsigma^2 - 1} = x$ so that $f(x) = e^{w_0 t\, x} - e^{-w_0 t\, x}$ and $g(x) = x$. We can calculate the derivatives with respect to x and examine the limit as $x \to 0$ as follows:

$$\frac{f(x)}{g(x)} = \frac{e^{w_0 t\, x} - e^{-w_0 t\, x}}{x}$$

$$\frac{f'(x)}{g'(x)} = \frac{w_0 t\, e^{w_0 t\, x} + w_0 t\, e^{-w_0 t\, x}}{1}$$

$$Limit_{x \to 0} \frac{f'(x)}{g'(x)} = 2w_0 t$$

Substituting this result for the bracketed term in [34] gives the expression for the current as a function of time, $i(t)$, in the critically damped case where $\varsigma = 1$.

$$\boxed{i(t) = \frac{V_{c0}}{R}\, e^{-w_0 t}\, 2w_0 t} \qquad [35]$$

Underdamped Solution

If the damping factor, ς, is less than 1 then the terms $\sqrt{\varsigma^2 - 1}$ in our general solution [31] result in the square root of a negative number and the values for s_1 and s_2 become complex conjugates (numbers having equal real parts and imaginary parts but with the same magnitude but opposite sign).

In electrical engineering solving the square root of negative numbers is facilitated by introducing the complex variable $j = \sqrt{-1}$ which denotes a unit vector on the imaginary axis of the complex plane (In mathematics the equivalent variable is *i*).

The utility of *j* is demonstrated as follows:

$j = \sqrt{-1}$ so $j^2 = -1$ therefore $\sqrt{-n} = \sqrt{j^2 n} = \sqrt{j^2}\sqrt{n} = j\sqrt{n}$

In the underdamped case $\varsigma < 1$ so:

$$\sqrt{\varsigma^2 - 1} = j\sqrt{1 - \varsigma^2} \qquad [36]$$

Substituting [36] directly into our general solution [31] gives:

$$i(t) = \frac{V_{c0}}{R} \frac{\varsigma}{j\sqrt{1-\varsigma^2}} \left[e^{-w_0\left(\varsigma - j\sqrt{1-\varsigma^2}\right)t} - e^{-w_0\left(\varsigma + j\sqrt{1-\varsigma^2}\right)t} \right] \qquad [37]$$

Collecting the complex terms (those involving *j*) from the real terms, again using the identity $e^{a+b} = e^a e^b$:

$$i(t) = \frac{V_{c0}}{R}\frac{\varsigma}{j\sqrt{1-\varsigma^2}}\left[e^{-w_0\varsigma t}e^{w_0 j\sqrt{1-\varsigma^2}t} - e^{-w_0\varsigma\, t}e^{-w_0 j\sqrt{1-\varsigma^2}t}\right]$$

$$= \frac{V_{c0}}{R}\frac{\varsigma}{\sqrt{1-\varsigma^2}}e^{-w_0\varsigma t}\left[\frac{e^{j\,w_0\sqrt{1-\varsigma^2}t} - e^{-j\,w_0\sqrt{1-\varsigma^2}t}}{j}\right] \qquad [38]$$

There are a number of identities equating complex exponentials to sines and cosines. The bracketed term in [38] is of the general form of one of these identities:

$$\frac{e^{j\beta} - e^{-j\beta}}{j} = 2\sin(\beta) \qquad \text{where } \beta = w_0\sqrt{1-\varsigma^2}t \qquad [39]$$

Substituting [39] into [38]] gives the expression for the current as a function of time, $i(t)$, in the under-damped case where $\varsigma < 1$.

$$\boxed{i(t) = \frac{V_{c0}}{R}\frac{\varsigma}{j\sqrt{1-\varsigma^2}}e^{-w_0\varsigma t}\, 2\sin\left(w_0\sqrt{1-\varsigma^2}t\right)} \qquad [40]$$

Conclusion

Equations [31], [35] and [40] can be used to estimate the current in your welder. These equations are easily extended to determine:

1. the voltage across the resistive elements in your discharge circuit, Vr, from [2].
2. the delivered power into a weld,
3. with slightly more mathematics, the voltage across the inductive elements, Vl, from [3].
4. the voltage across the capacitor bank, Vc, from [3].
5. with even more mathematics, the total delivered energy, $E = R_l \int_0^{\tau} i(t)^2\, dt$, and
6. the peak current in a given weld pulse, $\frac{di(t)}{dt} = 0$.

The accompanying spreadsheet completes these calculations (and a number of others) for you without any mathematics whatsoever.